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ON THE STRUCTURE OF C-BIFURCATION BOUNDARIES OF PIECEWISE-CONTINUOUS SYSTEMS

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Conformity is established between analytic conditions of C-bifurcation [1] and the spectrum of eigenvalues of matrices of point mapping linearized in the neighborhood of the degenerate trajectory. This approach allows to consider all possible basic structures of C-bifurcation boundaries of parameters space, and to reveal new cases of existence and birth of periodic modes, including the occurrence at C-bifurcations of attracting sets with diverging trajectories ("strange attractors"). Mathematical models are devised for the majority of structures of C-bifurcation boundaries. Some of the bifurcation transformations are obtained by one-dimensional mapping, which make it possible to follow closely the process of emergence of complex structures due to parameter variation.

1. Dynamic systems whose phase space consists of regions $\Phi_1, \ldots, \Phi_k$, merged in some way, in which motions are defined by different equations, are considered A characteristic of such systems is the specific sequence of passing through these regions. If at the change of parameter a section of the phase trajectory of periodic motion in Φ_i reaches the boundary of region Φ_j (a C-bifurcation), a further change of parameter results in a continuous transition of that motion to a mode of another type which contains one more trajectory section in Φ_j, and this corresponds to the simplest bifurcation pattern.

More complex situations are also possible. These include: merging of two motions of different types, followed by their disappearance, doubling of the oscillation period [1], emergence of a set of subharmonic modes [2] and of parameter space of extreme complexity in the neighborhood of the C-bifurcation boundary, producing the effect of boundary "blurring" [3]. (Complex motions whose analysis reduces to that of one-dimensional mapping were considered in, e.g., [4] and in the survey [5]). Investigation of complex motions in piecewise-continuous systems generally results in the analysis of discontinuous point mapping. Note that smooth systems of the Lorentz model type [6] may also lead to discontinuous mappings.

As formulated here, the phase trajectory L_0 of periodic motion that reaches one of the piecewise-continuous regions Φ_j corresponds to parameter $\mu = 0$. In the neighborhood of L_0 the trajectories generate a continuous point mapping Π of some surface D. Since the trajectories may either penetrate Φ_j or not, Π consists accordingly of maps Π^+ and Π^- merged into one.

We introduce on D a system of coordinates x_i such that the fixed point taken as the reference point would correspond to $\mu = 0$. We select the x_n-axis so that the matrices of equations linearized with respect to x_i and μ, which define transformations Π^+ and Π^-, differ by the elements of their last columns [1]. The equations of the linearized maps are then of the form

$$x' = Ax + c\mu, \quad x_n \geqslant 0 \tag{1.1}$$

$$x' = Bx + c\mu, \quad x_n \leqslant 0 \tag{1.2}$$

where x and c are n-dimensional vectors, μ is a small parameter, and A and B are square matrices in which $a_{ij} = b_{ij}$ when $j \neq n$. The latter ensures the continuity of the joint mapping (1.1), (1.2).

Note that multiple mapping of the type

$$\Pi = \begin{cases} \Pi^*\Pi^+, & x_n > 0 \\ \Pi^{**}\Pi^-, & x_n < 0 \end{cases}$$

is continuous when $\Pi^* = \Pi^{**}$ and discontinuous when $\Pi^* \neq \Pi^{**}$. Since among the four possible double mappings two are continuous

$$\Pi = \begin{cases} \Pi^+\Pi^+, & x_n \geqslant 0, \ x_n' \geqslant 0 \\ \Pi^+\Pi^-, & x_n \leqslant 0, \ x_n' \geqslant 0 \end{cases} \tag{1.3}$$

$$\Pi = \begin{cases} \Pi^-\Pi^+, & x_n \geqslant 0, \ x_n' \leqslant 0 \\ \Pi^-\Pi^-, & x_n \leqslant 0, \ x_n' \leqslant 0 \end{cases} \tag{1.4}$$

and two are discontinuous when $x_n, x_n' = 0$

$$\Pi = \begin{cases} \Pi^+\Pi^+, & x_n > 0, \ x_n' > 0 \\ \Pi^-\Pi^-, & x_n < 0, \ x_n' < 0 \end{cases} \tag{1.5}$$

$$\Pi = \begin{cases} \Pi^-\Pi^+, & x_n > 0, \ x_n' < 0 \\ \Pi^+\Pi^-, & x_n < 0, \ x_n' > 0 \end{cases} \tag{1.6}$$

To determine the character of dependence of periodic motions on μ, i.e. that of the C-bifurcation, we use the conditions based on the assumption of point mapping continuity [1]. Let $\chi_\alpha(\lambda)$ and $\chi_\beta(\lambda)$ be characteristic polynomials that correspond to fixed points x^* of mappings $x' = \Pi^+x$ and $x' = \Pi^-x$ at the limit of $\mu = 0$. Then at the change of μ one motion is converted into another, provided that condition

$$\chi_\alpha(1)\,\chi_\beta(1) > 0 \tag{1.7}$$

is satisfied.

Both periodic motions exist when either $\mu > 0$ or $\mu < 0$, and vanish after joining when $\mu = 0$, if

$$\chi_\alpha(1)\,\chi_\beta(1) < 0 \tag{1.8}$$

A motion of double period defined by the fixed points of mapping (1.6) is produced when condition

$$\chi_\alpha(-1)\,\chi_\beta(-1) < 0 \tag{1.9}$$

is satisfied.

Let us present conditions (1.7) - (1.9) in a form more convenient for considering possible structures of C-bifurcation boundaries. Let α_i and β_i $(i = 1, \ldots, n)$ be the spectra of eigenvalues of matrices A and B. We represent the characteristic polynomials in the form

$$\chi_\alpha(\lambda) = \prod_{i=1}^{n} (\lambda - \alpha_i), \quad \chi_\beta(\lambda) = \prod_{i=1}^{n} (\lambda - \beta_i) \tag{1.10}$$

Since the coefficients of these polynomials are real, the signs of expressions $\chi_\alpha(1)$ and $\chi_\beta(1)$ are determined by the number of real roots that exceed $+1$ (σ_α^+ and σ_β^+), respectively), while those of expressions $\chi_\alpha(-1)$ and $\chi_\beta(-1)$ depend on the number of real roots lying to the left of -1 (σ_α^- and σ_β^-). From this we have the following conditions.

C o n d i t i o n 1. If $\sigma_\alpha^+ + \sigma_\beta^+$ is even, then at the change of sign of μ the fixed point of transformation (1.1) continuously converts into the fixed point of transformation (1.2).

C o n d i t i o n 2. If $\sigma_\alpha^+ + \sigma_\beta^+$ is odd, the fixed points of transformations (1.1) and (1.2) lie on one side of the axis $\mu = 0$, merge at $\mu = 0$, and vanish on the other side.

C o n d i t i o n 3. If $\sigma_\alpha^- + \sigma_\beta^-$ is odd, a pair of fixed points of transformation (1.6) which corresponds to the motion of double period is generated.

To determine the location of the region of existence of the motion of double period relative to $\mu = 0$ it is sufficient to know the number $\overset{+}{\sigma}_{\alpha\beta} + \overset{+}{\sigma}_{\alpha\alpha}$ (or $\overset{+}{\sigma}_{\alpha\beta} + \overset{+}{\sigma}_{\beta\beta}$) of products of matrices AB and AA (or AB and BB) that exceed $+1$ and use Condition 1 or 2 for any of the continuous transformations (1.3) or (1.4). Note that the eigenvalues of matrices AA and BB are, respectively, equal α_i^2 and β_i^2 $(i = 1, \ldots, n)$.

2. From the entire manifold of structures of the parameter space bifurcation boundaries we separate the basic ones which we define as structures whose single and double periodic motions only have to be considered.

For convenience of representation of bifurcation transformations we use the following notation for periodic motions. The motion defined by a fixed point of mapping (1.1) is denoted by A if it is stable, and by a if unstable; the motion defined by a fixed point of mapping (1.2) correspondingly by B and b, and the motion defined by fixed points of the double transformation (1.6)

$$x' = BAx + (B + E)\,c\mu, \quad x_n > 0 \qquad (2.1)$$
$$x' = ABx + (A + E)\,c\mu, \quad x_n < 0$$

by AB and ab when these are stable and unstable, respectively. In this formula E denotes a unit matrix.

Since eigenvalues contained in a circle of unit radius correspond to stable periodic motions, the region of existence of any two of the three stable motions A, B, and AB can, in conformity with Condition 1, only lie on different sides of axis $\mu = 0$.

When at least one of the considered two periodic motions is unstable, generally any distribution of eigenvalues α_i, β_i, λ_i relative to $+1$ and -1 is formally possible. This implies the possibility of existence of seventeen basic structures of the C-bifurcation boundary, which represent five essentially different cases.

Case 1°. A simple change of the type of motion for which $\sigma_\alpha^+ + \sigma_\beta^+$ and $\sigma_\alpha^- + \sigma_\beta^-$ are even numbers. Depending on the specific values of α_i and β_i three transformation structures are possible

$$A \to B\ (n \geqslant 1), \quad A \to b\ (n \geqslant 2), \ a \to b\ (n \geqslant 1) \qquad (2.2)$$

where the dimension of point mapping with which the particular structure is possible is shown in parentheses.

Case 2°. Simple merging with disappearance of the two motion types for which $\sigma_\alpha^+ + \sigma_\beta^+$ is odd and $\sigma_\alpha^- + \sigma_\beta^-$ even. Two bifurcation transformation structures are possible

$$A,\ b \to \varnothing\ (n \geqslant 1),\ a,\ b \to \varnothing\ \ (n \geqslant 2) \qquad (2.3)$$

Case 3°. A change of the type of motion with formation of a double/oscillation/mode for which $\sigma_\alpha^- + \sigma_\beta^-$, is odd, while $\sigma_\alpha^+ + \sigma_\beta^+$ and $\sigma_{\alpha\beta}^+ + \sigma_{\alpha\alpha}^+$ are even. Five structures are possible in this case.

$$A \to b,\ AB\ (n \geqslant 1),\ A \to b,\ ab\ (n \geqslant 1),\ a \to B,\ ab\ (n \geqslant 1), \qquad (2.4)$$
$$a \to b,\ AB\ (n \geqslant 2),\ a \to b,\ ab\ (n \geqslant 2)$$

Case 4°. Merging with disappearance of the two types of motion and of the double oscillation mode. This occurs when all three numbers $\sigma_\alpha^+ + \sigma_\beta^+$, $\sigma_\alpha^- + \sigma_\beta^-$, and $\sigma_{\alpha\beta}^+ + \sigma_{\alpha\alpha}^+$ are odd. Three transformation structures are possible here

$$A,\ b,\ ab \to \varnothing\ (n \geqslant 2), \quad a,\ b,\ ab \to \varnothing\ (n \geqslant 1) \qquad (2.5)$$
$$a,\ b,\ AB \to \varnothing\ \ (n \geqslant 2)$$

Case 5°. Merging of two types of motion with formation of a double/oscillation/mode when $\sigma_\alpha^+ + \sigma_\beta^+$ and $\sigma_\alpha^- + \sigma_\beta^-$ are odd, and $\sigma_{\alpha\beta}^+ + \sigma_{\alpha\alpha}^+$ is even. Here four structures are formally possible

$$A,\ b \rightarrow AB, \quad A,\ b \rightarrow ab \qquad (2.6)$$
$$a,\ b \rightarrow AB, \quad a,\ b \rightarrow ab$$

The last case requires some clarification. The point is that in the case of transformation of a straight line into a straight line $(n = 1)$ the existence of a fixed double point implies the existence of a single one [4], i.e. the impossibility of realizing transformations (2.6). The impossibility of obtaining these structures for $n = 2$ can be proved, but the question of their realization for $n > 2$ remains for the time being open.

Formulas (2.2)—(2.6) show that twelve possible basic structures, in nine of which transformation is accompanied by stability loss, may correspond to C-bifurcations of stable periodic motions A, B, or AB.

3. It is shown below that it is always possible to obtain a mapping of the form (1.1), (1.2) that conform to specified spectra α_i and β_i.

Let $2n$ values of α_i and β_i be specified, and let it be required to determine elements a_{ij} and b_{ij} of matrices A and B. Since $a_{ij} = b_{ij}$ $(j \neq n)$, there are $n^2 + n$ unknowns. The problem is simplified by retaining $2n$ elements a_{in} and b_{in} only as the unknowns. Remaining elements can be specified with some degree of arbitrariness, but so as not to loose unknowns when writing down the principal minors. For example

$$a_{ij} = 0 \quad (i \neq j + 1) \quad a_{j+1,\,j} = 1 \quad (j = 1, 2, \ldots, n - 1) \qquad (3.1)$$

Coefficients γ_k of characteristic polynomials

$$\chi(\lambda) = \lambda^n + \gamma_1\lambda^{n-1} + \ldots + \gamma_n \qquad (3.2)$$

are equal to the sum, taken with factor $(-1)^k$, of all $\binom{n}{k}$ principal minors of the k-th order of the determinant det (A) or det (B) [7]. Since unknowns are only left in the last column, the above coefficients are linearly dependent on the sought elements. On the other hand, the definition of characteristic polynomials in (1.10) implies that γ_k are symmetric functions of specified roots

$$\gamma_k{}^\alpha = (-1)^k \sum_{i=1}^{n} \sum_{j=1}^{n} \ldots \sum_{r=1}^{n} \alpha_i\alpha_j \ldots \alpha_r \qquad (3.3)$$
$$\gamma_k{}^\beta = (-1)^k \sum_{i=1}^{n} \sum_{j=1}^{n} \ldots \sum_{r=1}^{n} \beta_i\beta_j \ldots \beta_r$$

where $i \neq j \neq \ldots \neq r$ over all factors k. As the result we obtain a system of $2n$ linear equations with a nonzero determinant.

As an example, we construct mappings (1.1), (1.2) for $n = 1, 2, 3$.

1) Two eigenvalues α and β are specified and taken as the elements of matrices of one-dimensional transformation. The eigenvalue of the matrix products AB is $\alpha\beta$.

2) Four eigenvalues α_1, α_2, β_1, and β_2 are specified. We seek matrices A and B of the form (3.1). We determine elements a_{12} and a_{22} by formulas

$$q_\alpha = \alpha_1\alpha_2 = \det(A) = -a_{12}, \quad p_\alpha = \alpha_1 + \alpha_2 = \mathrm{Tr}(A) = a_{22} \qquad (3.4)$$

We similarly obtain b_{12} and b_{22}. As the result, we have transformation matrices of the form

$$A = \begin{Vmatrix} 0 & -\alpha_1\alpha_2 \\ 1 & \alpha_1+\alpha_2 \end{Vmatrix}, \qquad B = \begin{Vmatrix} 0 & -\beta_1\beta_2 \\ 1 & \beta_1+\beta_2 \end{Vmatrix} \qquad (3.5)$$

Eigenvalues of the matrix product AB or of the similar product BA are determined by the roots of equation

$$\lambda^2 + (q_\alpha + q_\beta - p_\alpha p_\beta)\lambda + q_\alpha q_\beta = 0 \qquad (3.6)$$

3) Let us obtain matrices of three-dimensional transformation of the form (3.1) for specified spectra of α_1, α_2, α_3, β_1, β_2, and β_3. Elements a_{i3} are determined by formulas

$$\begin{aligned} \gamma_1^\alpha &= -a_{33} = -(\alpha_1 + \alpha_2 + \alpha_3) = -p_\alpha \\ \gamma_2^\alpha &= -a_{23} = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = s_\alpha \\ \gamma_3^\alpha &= -a_{13} = -\alpha_1\alpha_2\alpha_3 = -q_\alpha \end{aligned} \qquad (3.7)$$

Similarly we have

$$b_{13} = q_\beta, \quad b_{23} = -s_\beta, \quad b_{33} = p_\beta \qquad (3.8)$$

From (3.1), (3.7), and (3.8) we obtain the sough matrices

$$A = \begin{Vmatrix} 0 & 0 & q_\alpha \\ 1 & 0 & -s_\alpha \\ 0 & 1 & p_\alpha \end{Vmatrix}, \qquad B = \begin{Vmatrix} 0 & 0 & q_\beta \\ 1 & 0 & -s_\beta \\ 0 & 1 & p_\beta \end{Vmatrix} \qquad (3.9)$$

Eigenvalues of the matrix product AB are determined by the set of roots of equation

$$\lambda^3 + (s_\alpha + s_\beta - p_\alpha p_\beta)\lambda^2 + (s_\alpha s_\beta - p_\alpha q_\beta - p_\beta q_\alpha)\lambda - q_\alpha q_\beta = 0$$

4. The above analysis, limited to single and double periodic motions, only shows that, when the specified conditions are satisfied, the pattern of bifurction is not simpler than the one defined above as the basic. The complete bifurcation pattern may prove to be much more complex.

Of particular interest are C-bifurcations transformations with loss of stability and generation of several unstable modes, including the double mode. Appearance of the latter indicates the possibility of existence of even more complex periodic modes. Conditions under which these transformations are safe [8, 9] and all multiple fixed points unstable may correspond to the appearance of the strange attractor.

As the first example, let us consider the two-dimensional continuous mapping of the form (3.5) which depends on the three parameter μ, ρ, τ $(\rho, \tau > 0)$

$$x' = \frac{\rho\tau}{4} y, \quad y' = x + \frac{\tau-\rho}{2} y - \mu \quad (y \geqslant 0)$$

$$x' = \frac{1}{\rho\tau} y, \quad y' = x + \left(\frac{1}{\tau} - \frac{1}{\rho}\right) y - \mu \quad (y \leqslant 0)$$

In conformity with (3.4) and (3.5) the eigenvalues of transformation matrices are

$$\alpha_1 = \tau/2, \quad \alpha_2 = -\rho/2, \quad \beta_1 = 1/\tau, \quad \beta_2 = -1/\rho$$

For these eigenvalues Condition *1* is satisfied in the interval $1 < \tau < 2$, where one of the periodic modes exist for $\mu < 0$ and the other for $\mu > 0$. Condition *2* is satisfied in the region $0 < \tau < 1$ in which both modes exist for $\mu < 0$ and in region $\tau > 2$ where they exist for $\mu > 0$. Condition *3* is satisfied when $0 < \rho < 1$ and $\rho > 2$. A double periodic motion exists in these regions. The characteristic polynomial (3.6) of that motion assumes the form

$$\lambda^2 - \left(1 + \left(\frac{\rho}{2} - \frac{1}{\rho}\right)\left(\frac{\tau}{2} - \frac{1}{\tau}\right)\right)\lambda + \frac{1}{4} = 0$$

Motion AB is stable when the inequalities

$$-1 < \left(\frac{2}{\rho} - \rho\right)\left(\tau - \frac{2}{\tau}\right) < 9$$

are satisfied. Violation of the left-hand inequality corresponds to the appearance of root $\lambda = +1$, and that of the right-hand one of root $\lambda = -1$ which are shown in Fig. 1 by the solid and dash lines, respectively. Subdivision of the plane of parameters ρ and τ into regions with different structures of the C-bifurcation boundaries is also shown in Fig. 1, where the structures of regions denoted by the same numerals with and without primes differ by the direction of variation of parameter μ and by the periodic mode designation. For example, the increase of μ the

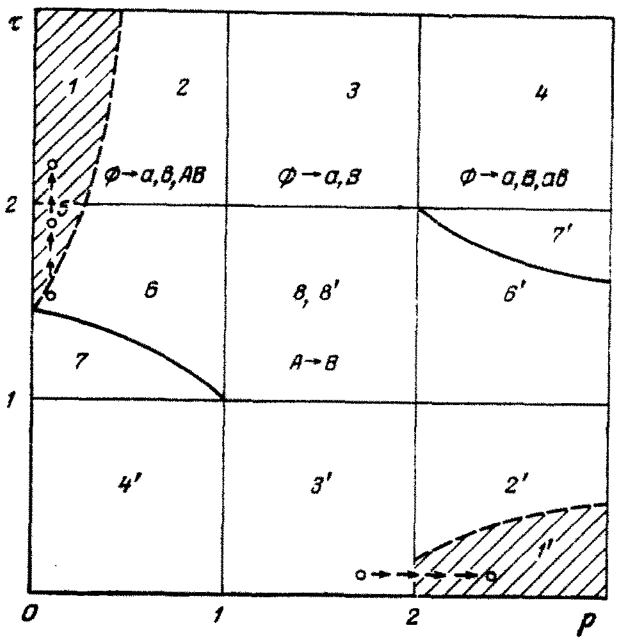

Fig. 1

structure of C-bifurcation transformation A, $ab \to b$ corresponds to region *7*, while transformation structure $a \to B$, ab corresponds to region *7'*. Regions in which existence of the strange attractor is possible are shaded. With allowance for (2.3) —(2.5) it is evident from Fig. 1 that the separated regions can be reached by using a sequence of bifurcation transformations obtained even by one-dimensional point mapping. These transformations are shown by arrows. Thus, to the transformation due to change of parameter τ, first, corresponds structure $A \to b$, AB (region *6*), then structure $A \to b$, ab (region *5*), and finally structure $\varnothing \to a$, b, ab (region *1*).

Occurence of strange attractors is examined on the example of one-dimensional mapping

$$x' = \alpha x - \mu, \quad x \geqslant 0 \qquad (4.1)$$
$$x' = \beta x - \mu, \quad x \leqslant 0 \quad (\alpha > 0, \beta < 0)$$

When $\mu < 0$ and $\alpha < 1$ there exists a stable fixed point $x^* = \mu / (\alpha - 1)$ of mode A, while for $\mu > 0$ and $\beta < -1$ we always have the unstable point $x^* = \mu / (\beta - 1)$ of mode b and a pair of fixed point of the double mode. Fixed point of any multiplicity are evidently unstable when $\alpha > 1$ and $\beta < -1$.

By an appropriate selection of parameters $\alpha < 1$ and $\beta < -1$ it is possible to obtain a C-bifurcation transformation with which the stable mode A vanishes at $\mu = 0$ simultaneously generating not less than a set of periodic modes b, $ab, \ldots, a^{k-2}b, A^{k-1}B$ $(k \geqslant 2)$ of which the most complex one is stable.

The conditions of existence of mode $a^{k-1}b$ reduce to conditions for the signs of the fixed point sequence

$$x_1^* \leqslant 0, \quad x_2^* \geqslant 0, \ldots, \quad x_k^* \geqslant 0 \tag{4.2}$$

Expressing the values of these points in terms of μ, α, and β, we readily find that all of conditions (4.2) are satisfied for $\mu > 0$, if

$$\beta < -(1 + \alpha^{-1} + \alpha^{-2} + \ldots + \alpha^{-k+2}) = f_k(\alpha) \tag{4.3}$$

It follows from (4.3) that when α and β are specified and there exists a mode of multiplicity l all modes of lower multiplicity exist, since $f_l(\alpha) < f_k(\alpha)$ for $k < l$.

Since the eigenvalue of the mode of k-multiplicity is $\alpha^{k-1}\beta$, the condition of that mode stability is of the form

$$\beta > -\alpha^{-k+1} = \varphi_k(\alpha) \tag{4.4}$$

Analysis of conditions (4.3) and (4.4) shows that with decreasing β the mode of multiplicity k first becomes unstable, and only after that a mode of multiplicity $k+1$ is generated, as well as that for any $k \geqslant 2$ there exist α and β which simultaneously satisfy both conditions

$$\varphi_k(\alpha) < \beta < f_k(\alpha) \tag{4.5}$$

since the remainder of the monotonically increasing functions φ_k and f_k changes its sign and the functions intersect at some point $0 < \alpha_k < 1$.

The above statement is proved. When (4.5) is satisfied, then at $\mu = 0$ we have the safe transformation $A \to A^{k-1}B, \ldots$. Note that in the case of the revealed set of structures the concept of safe and unsafe C-bifurcation boundary cannot be uniquely linked with its simplest structure, as was done in [1]. When for $\mu > 0$ there are not stable fixed points, the transformation may be either unsafe (e.g., $A \to b$) or safe. Regions of possible existence of an attractor is shaded in Fig. 2 where the dash line denotes the boundary of the region of appearance of triple fixed points; above that line modes of any multiplicity can exist [4].

A Lamery diagram for five points of the parameter plane is shown in Fig. 3, where the notation conforms to Fig. 2 $(\mu > 0)$. The diagram lower part shows the nominal subdivision of the x-axis into intervals of "contraction" and "dispersal" of points of mapping (4.1). The unstable simple fixed point x_1 (black dots) and the mapped onto it point x_0 (small circles) define in Fig. 3 the subdivision boundaries

$$x_1 = \frac{\mu}{\alpha - 1}, \quad x_0 = \frac{\mu\alpha}{\beta(\alpha - 1)}$$

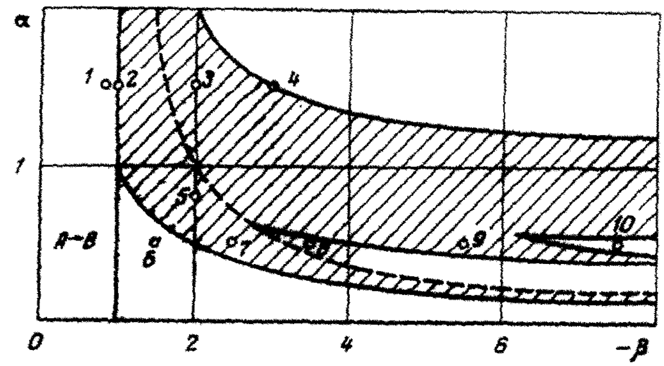

Fig. 2

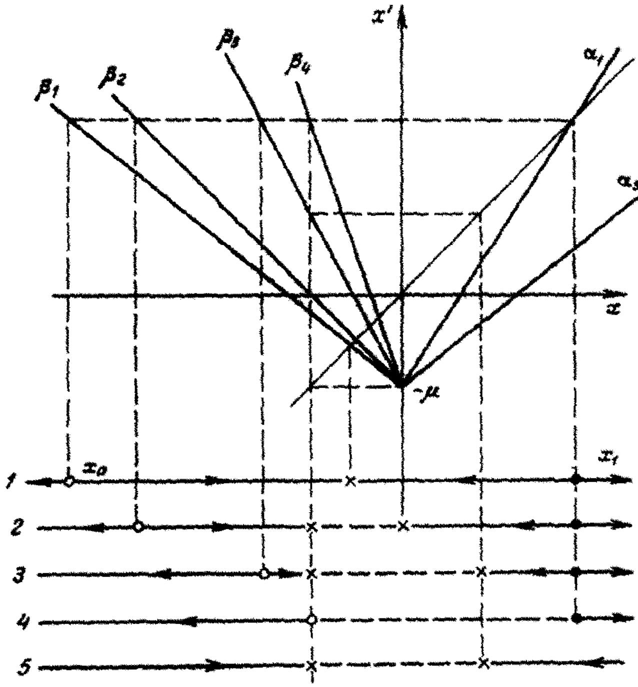

Fig. 3

When $-1 < \beta < 0$ (Case 1 of α_1 and β_1 with the related structure $\varnothing \to a, B$ in Figs. 2 and 3), the interval x_0x_1 contains a single stable fixed point. Case 2 (α_1, β_2) corresponds to transition through the stability boundary $\beta = -1$ at which a continuum of double fixed points $-\mu < x^*_{1,2} < 0$ is generated. With decreasing β^*(Case 3 of α_1 and β_3 with related structure $\varnothing \to a, b, ab, \ldots, \infty$) the mapping points in the interval x_0x_1 lead to the attraction interval $-\mu < x < -\mu(1+\beta)$ (shown in Fig. 3 by the dash line) inside which stable fixed points cannot exist. With further decrease of β points x_0 and x_1 approach the attraction interval, merging with its ends when the image of point $x = 0$ is at point x_0. Case 4 (α_1, β_4) has been chosen on the bifurcation boundary $\alpha + \alpha\beta - \beta = 0$ which corresponds to the loss of the attraction property by the interval.

Cases 1 and 3 relate to an unsafe C-boundary: for $\mu > 0$ there exists either a stable fixed point or a strange attractor both of which vanish at $\mu = 0$. The stable point merges with the unstable and, when the strange attractor vanishes, the position of the attraction interval within the contraction interval remains unchanged for any as small as desired $\mu > 0$.

For $\alpha < 1$ the attraction interval becomes "absolutely stable" (Case 5 of α_5, β_5 with structure $A \to b, ab$). The C-boundary $\mu = 0$ is here safe and corresponds the transformation of motion A either into a stable periodic mode of complex type or into a strange attractor. Computer simulation of mapping (4.1) showed convergence to the respective periodic sequence of fixed points (Case 6 with the related structure $A \to b, AB$, Case 8 and structure $A \to A^2B, \ldots$, and Case 10 with structure $A \to A^3\bar{B}, \ldots$, shown in Fig. 2) or the complete filling of the attraction interval (Case 5, Case 7, with $A \to b, ab$, and Case 9 with $A \to b, ab, \ldots, \infty$).

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